ON THE ASYMPTOTIC NORMALITY OF CERTAIN NONPARAMETRIC TEST STATISTICS *

關於某些無母數檢定統計量的漸近常態性之研究

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ABSTRACT

Suppose $(X_{11},X_{21}),\ldots,(X_{1m},X_{2m})$ and $(Y_{11},Y_{21}),\ldots,(Y_{1n},Y_{2n})$ are two independent bivariate random samples from populations with continuous distribution functions $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ respectively. We assume that the two populations have a common mean, which is either known or unknown. We would like to detect differences in variability or dispersion for the two populations. However, the bivariate case seems to have been studied far less fully than a univariate one. In this paper, we suggest two nonparametric tests W and W* and establish the asymptotic normality of W and W* under fairly general conditions on the underlying distribution functions $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$. This asymptotic property is very useful in investigating the efficiency of the test procedures.

摘 要

假定(X_{11},X_{21})，…，(X_{1m},X_{2m})及(Y_{11},Y_{21})，…，(Y_{1n},Y_{2n})是分別從具有連續分配函數$F_{X_1,X_2}(x_1,x_2)$及$G_{Y_1,Y_2}(y_1,y_2)$之母體抽取出來的兩個獨立二元(bivariate)隨機樣本；我們假設這兩個母體具有相同之平均數(mean)，而這平均數可以是已知，也可以是未知。我們想要測出這兩個母體之離勢(variability or dispersion)是否不同。母體是一元(univariate)的情況已有甚多的研究發表。然而像這類母體是二元的情況，至今甚少研究。在這篇論文中，提出了兩種無母數檢定 W及W*，並且在連續分配$F_{X_1,X_2}(x_1,x_2)$及$G_{Y_1,Y_2}(y_1,y_2)$滿足相當一般的條件下，可得出 W及W*的漸近常態性。這種漸近性在研究檢定過程的有效性(efficiency)是極爲有用的。

I. INTRODUCTION

Suppose $(X_{11}, X_{21}), \ldots, (X_{1m}, X_{2m})$ and $(Y_{11}, Y_{21}), \ldots, (Y_{1n}, Y_{2n})$ are two independent bivariate random samples from populations with continuous distribution functions $F_{X_1,X_2}(x_1,x_2)$, and $G_{Y_1,Y_2}(y_1,y_2)$ respectively. We assume that the two populations have a common mean, which is either known or unknown. We would like to detect differences in variability or dispersion for the two populations.

It is conceivable that the sample points from the population which has a larger dispersion will tend to be further away from the common mean. This suggests us to define $W_{m,n}$ to be the Mann-Whitney test statistic for the distances from the sample points to the common mean if it is known, and to define $W^*_{m,n}$ by using the combined sample mean instead of the common mean if it is unknown.

Under the null hypothesis H_0: F = G, the distributions of W and W* are the same as that of the Mann-Whitney test statistic. Hence both tests are nonparametric.

The asymptotic normality of $W_{m,n}$ for any fixed $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ can be obtained by applying the Chernoff-Savage theorem and the fact that the Mann-Whitney test and the Wilcoxon rank sum test are equivalent. In this paper, we prove the asymptotic normality of $W^*_{m,n}$ under fairly general conditions on the underlying distributions $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$. This asymptotic property is very useful in investigating the efficiency of the test procedures.

II. NOTATIONS

Set N = m + n.

For i = 1, . . . , m and j = 1, . . . , n, define

$$U^*_{iN} = [(X_{1i} - \bar{Z}_{1N})^2 + (X_{2i} - \bar{Z}_{2N})^2]^{1/2},$$

$$V^*_{jN} = [(Y_{1j} - \bar{Z}_{1N})^2 + (Y_{2j} - \bar{Z}_{2N})^2]^{1/2},$$

$$U_i = [(X_{1i} - \mu_1)^2 + (X_{2i} - \mu_2)^2]^{1/2}, \text{ and}$$

$$V_j = [(Y_{1j} - \mu_1)^2 + (Y_{2j} - \mu_2)^2]^{1/2},$$

where $\bar{Z}_{1N} = (X_{11} + \ldots + X_{1m} + Y_{11} + \ldots + Y_{1n})/N$,

$\bar{Z}_{2N} = (X_{21} + \ldots + X_{2m} + Y_{21} + \ldots + Y_{2n})/N$, and

$\underset{\sim}{\mu} = (\mu_1, \mu_2)$ is the true common mean.

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We define $W^*_{m,n}$ to be the Mann-Whitney test statistic for the two samples,

$$U^*_{1N}, U^*_{2N}, \ldots, U^*_{mN} \text{ and } V^*_{1N}, V^*_{2N}, \ldots, V^*_{nN}$$

i.e.,
$$W^*_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} D^*_{ij},$$

where
$$D^*_{ij} = 1 \quad \text{if } U^*_{iN} > V^*_{jN} \quad \text{for all } i = 1, 2, \ldots, m,$$
$$= 0 \quad \text{otherwise} \quad j = 1, 2, \ldots, n,$$

and $W_{m,n}$ to be the Mann-Whitney test statistic for the two samples,

$$U_1, U_2, \ldots, U_m \text{ and } V_1, V_2, \ldots, V_n$$

i.e.,
$$W_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} D_{ij},$$

where
$$D_{ij} = 1 \quad \text{if } U_i > V_j \quad \text{for all } i = 1, 2, \ldots, m,$$
$$= 0 \quad \text{otherwise} \quad j = 1, 2, \ldots, n.$$

III. ASYMPTOTIC NORMALITY

Applying the techniques used in Chernoff and Savage (1958), Raghavachari (1965), and Fligner (1974), we show that the limiting distribution of $W^*_{m,n}$ is the same as that of $W_{m,n}$ under fairly general conditions on the underlying distribution $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$, namely that

(i) $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ are absolutely continuous with p.d.f.'s $f(x_1,x_2)$ and $g(y_1,y_2)$ respectively,

(ii) $f(x_1,x_2)$ and $g(y_1,y_2)$ are bounded and their partial derivatives $f_1(x_1,x_2)$, $f_2(x_1,x_2)$, $g_1(y_1,y_2)$ and $g_2(y_1,y_2)$ are continuous.

(iii) $f(x_1,x_2)$ and $g(y_1,y_2)$ are symmetric about the common mean (μ_1,μ_2), i.e.,
for each (x_1,x_2), $f(\mu_1 + x_1, \mu_2 + x_2) = f(\mu_1 - x_1, \mu_2 - x_2)$ and
for each (y_1,y_2), $g(\mu_1 + y_1, \mu_2 + y_2) = g(\mu_1 - y_1, \mu_2 - y_2)$.

(iv) The p.d.f.'s of $U = [(X_1 - \mu_1)^2 + (X_2 - \mu_2)^2]^{1/2}$ and $V = [(Y_1 - \mu_1)^2 + (Y_2 - \mu_2)^2]^{1/2}$, say $s(u)$ and $t(v)$ respectively, are bounded and continuous.

It can easily be cnecked that these conditions are satisfied by the usual distributions such as bivariate normal, bivariate uniform and bivariate t, etc.

Theorem 3.1.: Let $(X_{11},X_{21}), \ldots (X_{1m},X_{2m})$ and $(Y_{11},Y_{21}), \ldots, (Y_{1n},Y_{2n})$ be two independent bivariate random samples from populations with continuous d.f.'s $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ respectively. If $F_{X_1,X_2}(x_1,x_2)$ and $G_{Y_1,Y_2}(y_1,y_2)$ satisfy the four conditions stated above, then $W_{m,n}$ and $W^*_{m,n}$ have the same limiting distribution, which implies that $\frac{W^*_{m,n} - E(W_{m,n})}{\sqrt{Var(W_{m,n})}}$ is asymptotically normal, providing $Var(W_{m,n}) \neq 0$, and m/N remains constant.

Proof: We first need several lemmas.

Lemma *3.2.*: Let $\{R_{m,n}\}$ and $\{R^*_{m,n}\}$ be two sequences of random variables such that $|R_{m,n} - R^*_{m,n}| \xrightarrow[m,n \to \infty]{P} 0$ and $R_{m,n} \xrightarrow[m,n \to \infty]{L} R$, where R is a random variable.

Then $R^*_{m,n} \xrightarrow[m,n \to \infty]{L} R$.

Proof: See {Rao (1965), p. 101}.

Lemma *3.3.*: Suppose $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m$ are random vectors with joint d.f. $F(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m ; \underset{\sim}{\nu})$ where $\underset{\sim}{\nu}$ is some paramater vector. Let $\hat{\underset{\sim}{\nu}}(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m)$ be an estimate of $\underset{\sim}{\nu}$ such that $\sqrt{m}(\hat{\underset{\sim}{\nu}} - \underset{\sim}{\nu})$ is bounded in probability (i.e., Given $\varepsilon > 0$, there exists a number b_o and a positive integer M_o such that $P(|\hat{\underset{\sim}{\nu}} - \underset{\sim}{\nu}| \geqslant b_o/\sqrt{m}) < \varepsilon$ for all $m \geqslant M_o$). Suppose $T_m(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m ; \underset{\sim}{\nu})$ is a sequence of statistics. If $\sup_{|\underset{\sim}{t} - \sqrt{m}\underset{\sim}{\nu}| \leqslant c} |T_m(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m ; \underset{\sim}{t}/\sqrt{m})| \xrightarrow[m \to \infty]{P} 0$ for any c, then $T_m(\underset{\sim}{X}_1, \ldots \underset{\sim}{X}_m ; \hat{\underset{\sim}{\nu}}) \xrightarrow[m \to \infty]{P} 0$.

Proof: This is an extension of Lemma 3.1 in Fligner (1974). Both proofs are essentially the same.

Lemma *3.4.*: Let $\overline{\underset{\sim}{Z}}_N = (\overline{Z}_{1N},\overline{Z}_{2N})$, where

$$\overline{Z}_{1N} = \frac{X_{11} + X_{12} + \ldots + X_{1m} + Y_{11} + Y_{12} + \ldots + Y_{1n}}{N}, \text{ and}$$

$$\overline{Z}_{2N} = \frac{X_{21} + X_{22} + \ldots + X_{2m} + Y_{21} + Y_{22} + \ldots + Y_{2n}}{N}, \text{ and}$$

$\underset{\sim}{\mu} = (\mu_1, \mu_2)$. Then $\sqrt{N}(\overline{\underset{\sim}{Z}}_N - \underset{\sim}{\mu})$ is bounded in probability.

Proof: Let $\varepsilon > 0$ be given. Take $b_1 > (2[Var(X_1) + Var(Y_1)]/\varepsilon)^{1/2}$. Then, we have $P(|\overline{Z}_{1N} - \mu_1| \geqslant b_1/\sqrt{N}) < \varepsilon/2$ for all N, by Chebyshev's inequality. Similarly, there exists a number b_2 such that $P(|\overline{Z}_{2N} - \mu_2| \geqslant b_2/\sqrt{N}) < \varepsilon/2$ for all

N.

Take $b = 2\max(b_1, b_2)$. Then

$$P(|\bar{Z}_N - \mu| \geqslant b/\sqrt{N}) \leqslant P(|\bar{Z}_{1N} - \mu_1| + |\bar{Z}_{2N} - \mu_2| \geqslant b/\sqrt{N})$$
$$\leqslant P(|\bar{Z}_{1N} - \mu_1| \geqslant [\max(b_1, b_2)]/\sqrt{N}) + P(|\bar{Z}_{2N} - \mu_2| \geqslant [\max(b_1, b_2)]/\sqrt{N})$$
$$\leqslant P(|\bar{Z}_{1N} - \mu_1| \geqslant b_1/\sqrt{N}) + P(|\bar{Z}_{2N} - \mu_2| \geqslant b_2/\sqrt{N})$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ for all N.}$$

Hence $\sqrt{N}(\bar{Z}_N - \mu)$ is bounded in probability.

Without loss of generality, we can take $\mu_1 = \mu_2 = 0$. Now, fix t_1 and t_2. For $i=1, \ldots, m$ and $j=1, \ldots, n$, define

$U_i = (X_{1i}^2 + X_{2i}^2)^{1/2}$, $\qquad V_j = (Y_{1j}^2 + Y_{2j}^2)^{1/2}$,

$U_{iN}^{**} = [(X_{1i} - t_1/\sqrt{N})^2 + (X_{2i} - t_2/\sqrt{N})^2]^{1/2}$, $V_{jN}^{**} = [(Y_{1j} - t_1/\sqrt{N})^2 + (Y_{2j} - t_2/\sqrt{N})^2]^{1/2}$,

$S_m(x) = [\text{number of } (X_{1i}, X_{2i}) \text{ such that } U_i \leqslant x]/m$,

$T_n(x) = [\text{number of } (Y_{1j}, Y_{2j}) \text{ such that } V_j \leqslant x]/n$,

$H_N(x) = \lambda_N S_m(x) + (1 - \lambda_N) T_n(x)$, where $\lambda_N = m/N$,

$S_m^{**}(x) = [\text{number of } (X_{1i}, X_{2i}) \text{ such that } U_{iN}^{**} \leqslant x]/m$,

$T_n^{**}(x) = [\text{number of } (Y_{1j}, Y_{2j}) \text{ such that } V_{jN}^{**} \leqslant x]/n$, and

$H_N^{**}(x) = \lambda_N S_m^{**}(x) + (1 - \lambda_N) T_n^{**}(x)$. Define also

$$T_N = \frac{1}{m} \sum_{i=1}^{N} J_N(i/N) Z_{Ni}, \qquad T_N' = \frac{1}{m} \sum_{i=1}^{N} J_N(i/N) Z_{Ni}^{**}$$

where $J_N(i/N) = i/N$, $Z_{Ni} = 1$ if the i^{th} smallest of N observations is a U

$= 0$ otherwise,

$Z_{Ni}^{**} = 1$ if the i^{th} smallest of N observations is a U^{**}

$= 0$ otherwise, and

$S(x)$ and $T(x)$ are d.f.'s of U and V respectively.

As in Chernoff and Savage (1958), T_N and T_N' have the following integral representation

$$T_N = \int_0^\infty J_N(H_N(x))dS_m(x), \qquad T'_N = \int_0^\infty J_N(H_N^{**}(x))dS_m^{**}(x).$$

Throughout our proofs K will denote a generic constant which will not depend on m,n,N. Let I_N be the (random) interval in which $0 < H_N(x) < 1$, and I_N^{**} be the (random) interval in which $0 < H_N^{**}(x) < 1$, and throughout this section let $J_N(x) = J(x) = x$.

Proceeding as in Chernoff and Savage (1958) and Raghavachari (1965), we can write

$$T'_N = \int_0^\infty J_N(H_N^{**}(x))dS_m^{**}(x)$$

$$= \int_{0<H_N^{**}<1} [J_N(H_N^{**}) - J(H_N^{**})]\, dS_m^{**}(x) + \int_{0<H_N^{**}<1} J(H_N^{**})dS_m^{**}(x).$$

In the second integral on the right write

$$dS_m^{**} = d(S_m^{**} - S + S),$$

$$J(H_N^{**}) = J(H) + (H_N^{**} - H)J'(H) + \frac{(H_N^{**} - H)^2}{2} J'' \quad \gamma H_N^{**} + (1-\gamma)H\}, 0 < \gamma < 1.$$

After multiplying out, the expression becomes

$$T'_N = A^{**} + B_{1N}^{**} + B_{2N}^{**} + \sum_{i=1}^{6} C_{iN}^{**}$$

where

$$A^{**} = \int_{0<H<1} J(H)dS(x),$$

$$B_{1N}^{**} = \int_{0<H<1} J(H)d[S_m^{**}(x) - S(x)],$$

$$B_{2N}^{**} = \int_{0<H<1} (H_N^{**} - H)J'(H)dS(x),$$

$$C_{1N}^{**} = \lambda_N \int_{0<H<1} (S_m^{**} - S)J'(H)d(S_m^{**}(x) - S(x)),$$

$$C_{2N}^{**} = (1-\lambda_N) \int_{0<H<1} (T_n^{**} - T)J'(H)d(S_m^{**}(x) - S(x)),$$

$$C_{3N}^{**} = \int_{I_N^{**}} \frac{(H_N^{**} - H)^2}{2} J''(\gamma H_N^{**} + (1-\gamma)H)dS_m^{**}(x), 0 < \gamma < 1,$$

$$C_{4N}^{**} = \int_{H_N^{**}=1} (-J(H) - (H_N^{**} - H)J'(H))dS_m^{**}(x),$$

$$C_{5N}^{**} = \int_{I_N^{**}} (J_N(H_N^{**}) - J(H_N^{**}))dS_m^{**}(x),$$

$$C_{6N}^{**} = \int_{H_N^{**}=1} J_N(H_N^{**})dS_m^{**}(x).$$

Lemma 3.5.: For every t_1, t_2, we have $\sqrt{N}(T_N' - T_N) \xrightarrow[N\to\infty]{P} 0$, which implies that $S_N(\underset{\sim}{t}) = m^{-\frac{1}{2}}n^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\{\Phi(\underset{\sim}{X}_i - \underset{\sim}{t}/\sqrt{N}, \underset{\sim}{Y}_j - \underset{\sim}{t}/\sqrt{N}) - \Phi(\underset{\sim}{X}_i, \underset{\sim}{Y}_j)\} \xrightarrow[N\to\infty]{P} 0$,
where $\Phi(\underset{\sim}{x},\underset{\sim}{y}) = 1$ if $|\underset{\sim}{x}| > |\underset{\sim}{y}|$
$= 0$ otherwise.

Proof: A^{**} is the same as A of Chernoff and Savage (1958), and is finite. We need to show that for fixed λ_N,

$$\sqrt{N}(B_{1N}^{**} + B_{2N}^{**}) - \sqrt{N}(B_{1N} + B_{2N}) \xrightarrow[N\to\infty]{P} 0, \text{ and}$$

$$\sqrt{N}C_{iN}^{**} \xrightarrow[N\to\infty]{P} 0 \text{ for } i = 1,2,3,4,5,6.$$

(1) Claim $\sqrt{N}(B_{1N}^{**} + B_{2N}^{**}) - \sqrt{N}(B_{1N} + B_{2N}) \xrightarrow[N\to\infty]{P} 0$.

$$B_{1N}^{**} = \int_{0<H<1} J(H)d[S_m^{**}(x) - S(x)] = \int_{0<H<1} Hd[S_m^{**}(x) - S(x)],$$

since $J(H) = H$,

$$B_{2N}^{**} = \int_{0<H<1} (H_N^{**} - H)J'(H)dS(x) = \int_{0<H<1} (H_N^{**} - H)dS(x), \text{ since } J'(H)=1.$$

Integrating B_{2N}^{**} by parts and using the fact that $\int_0^\infty d(H_N^{**} - H) = 0$,
we obtain

$$\begin{aligned} B_{1N}^{**} + B_{2N}^{**} &= \int_0^\infty [\lambda_N S(x) + (1-\lambda_N)T(x)]\, d[S_m^{**}(x) - S(x)] \\ &\quad - \int_0^\infty \lambda_N S(x) d[S_m^{**}(x) - S(x)] \\ &\quad - \int_0^\infty (1-\lambda_N)S(x)d[T_n^{**}(x) - T(x)] \\ &= (1-\lambda_N)\{\int_0^\infty T(x)d[S_m^{**}(x) - S(x)] - \int_0^\infty S(x)d[T_n^{**}(x) - T(x)]\} \\ &= (1-\lambda_N)\{m^{-1}\sum_{i=1}^{m}[T(U_{iN}^{**}) - E(T(U))] - n^{-1}\sum_{j=1}^{n}[S(V_{jN}^{**}) - E(S(V))]\}. \end{aligned}$$

Similarly, $B_{1N} + B_{2N} = (1-\lambda_N)\{m^{-1}\sum_{i=1}^{m}[T(U_i) - E(T(U))]$

$$- n^{-1}\sum_{j=1}^{n}[S(V_j) - E(S(V))]\}.$$

Hence $\sqrt{N}(B_{1N}^{**}+B_{2N}^{**})-\sqrt{N}(B_{1N}+B_{2N})$

$$=\sqrt{N}(1-\lambda_N)\{m^{-1}\sum_{i=1}^{m}[T(U_{iN}^{**})-T(U_i)]-n^{-1}\sum_{j=1}^{n}[S(V_{jN}^{**})-S(V_j)]\}.$$

Let $L_{N,t}=1/\sqrt{N}\sum_{i=1}^{m}[T(U_{iN}^{**})-T(U_i)]$

$$=1/\sqrt{N}\sum_{i=1}^{m}[M(X_{1i}-t_1/\sqrt{N},X_{2i}-t_2/\sqrt{N})-M(X_{1i},X_{2i})],$$

where $M(x,y)=T[(x^2+y^2)^{1/2}]$,

$$=\lambda_N m^{-1}\sum_{i=1}^{m}\sqrt{N}[M(X_{1i}-t_1/\sqrt{N},X_{2i}-t_2/\sqrt{N})-M(X_{1i},X_{2i})].$$

Note that $\sqrt{N}[M(X_{1i}-\frac{t_1}{\sqrt{N}},X_{2i}-\frac{t_2}{\sqrt{N}})-M(X_{1i},X_{2i})]$

$$=\sqrt{N}[-\frac{t_1}{\sqrt{N}}M_1(X_{1i}-\frac{\xi_i t_1}{\sqrt{N}},X_{2i}-\frac{\xi_i t_2}{\sqrt{N}})-\frac{t_2}{\sqrt{N}}M_2(X_{1i}-\frac{\xi_i t_1}{\sqrt{N}},X_{2i}-\frac{\xi_i t_2}{\sqrt{N}})]$$

by the Mean-Value Theorem for several variables,
where $0<\xi_i<1$,

$$M_1(x,y)=\frac{\partial M(x,y)}{\partial x}=x(x^2+y^2)^{-1/2}t((x^2+y^2)^{1/2}),\text{ and}$$

$$M_2(x,y)=\frac{\partial M(x,y)}{\partial y}=y(x^2+y^2)^{-1/2}t((x^2+y^2)^{1/2}),$$

$$=-t_1M_1(X_{1i}-\frac{\xi_i t_1}{\sqrt{N}},X_{2i}-\frac{\xi_i t_2}{\sqrt{N}})-t_2M_2(X_{1i}-\frac{\xi_i t_1}{\sqrt{N}},X_{2i}-\frac{\xi_i t_2}{\sqrt{N}}).$$

By the assumption that g is bounded, the p.d.f.

$$t(v)=v\int_0^{2\pi}g(v\cos\theta,v\sin\theta)d\theta\quad\text{for }0<v<\infty$$
$$\leqslant Kv.$$

Hence $|M_1(x,y)|=|x(x^2+y^2)^{-1/2}t((x^2+y^2)^{1/2})|\leqslant K|x|$, and

$|M_2(x,y)|=|y(x^2+y^2)^{-1/2}t((x^2+y^2)^{1/2})|\leqslant K|y|$, which implies

that $|\sqrt{N}[M(X_{1i}-\frac{t_1}{\sqrt{N}},X_{2i}-\frac{t_2}{\sqrt{N}})-M(X_{1i},X_{2i})]|$

$$\leqslant|t_1||M_1(X_{1i}-\frac{\xi_i t_1}{\sqrt{N}},X_{2i}-\frac{\xi_i t_2}{\sqrt{N}})|+|t_2||M_2(X_{1i}-\frac{\xi_i t_1}{\sqrt{N}},X_{2i}-\frac{\xi_i t_2}{\sqrt{N}})|$$

$$\leqslant K(|X_{1i}|+|t_1|)+K(|X_{2i}|+|t_2|).$$

It follows that by an application of Chebyshev's inequality,

$$P(|m^{-1}\sum_{i=1}^{m}\{\sqrt{N}[M(X_{1i}-\tfrac{t_1}{\sqrt{N}},X_{2i}-\tfrac{t_2}{\sqrt{N}})-M(X_{1i},X_{2i})]-E(\sqrt{N}[M(X_{1i}-\tfrac{t_1}{\sqrt{N}},X_{2i}-\tfrac{t_2}{\sqrt{N}})$$

$$-M(X_{1i},X_{2i})])\}|\geqslant \varepsilon)$$

$$\leqslant \frac{\sum_{i=1}^{m} E(\{\sqrt{N}[M(X_{1i}-\frac{t_1}{\sqrt{N}},X_{2i}-\frac{t_2}{\sqrt{N}})-M(X_{1i},X_{2i})]\}^2)}{m^2\varepsilon^2}$$

$$\leqslant \frac{mE([K(|X_1|+|t_1|)+K(|X_2|+|t_2|)]^2)}{m^2\varepsilon^2} \xrightarrow[N\to\infty]{} 0.$$

Also, by the dominated convergence theorem and the assumption that t is continuous,

$$E(\sqrt{N}[M(X_{1i}-\tfrac{t_1}{\sqrt{N}},X_{2i}-\tfrac{t_2}{\sqrt{N}})-M(X_{1i},X_{2i})])$$

$$=E(-t_1M_1(X_{1i}-\tfrac{\xi_i t_1}{\sqrt{N}},X_{2i}-\tfrac{\xi_i t_2}{\sqrt{N}})-t_2M_2(X_{1i}-\tfrac{\xi_i t_1}{\sqrt{N}},X_{2i}-\tfrac{\xi_i t_2}{\sqrt{N}}))$$

$$\xrightarrow[N\to\infty]{} -t_1E(M_1(X_{1i},X_{2i}))-t_2E(M_2(X_{1i},X_{2i})).$$

Further,

$$E(M(X_{1i},X_{2i}))=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1(x_1^2+x_2^2)^{-1/2}t((x_1^2+x_2^2)^{1/2})f(x_1,x_2)dx_1dx_2$$

$= 0$, by the assumption that f is symmetric about the mean (μ_1,μ_2).

Similarly, $E(M_2(X_{1i},X_{2i}))=0$.

Therefore, $m^{-1}\sum_{i=1}^{m}\sqrt{N}[M(X_{1i}-\frac{t_1}{\sqrt{N}},X_{2i}-\frac{t_2}{\sqrt{N}})-M(X_{1i},X_{2i})]\xrightarrow[N\to\infty]{P} 0.$

Accordingly, $L_{N,\underset{\sim}{t}}\xrightarrow[N\to\infty]{P} 0.$

In a similar manner we can show that

$\frac{1}{\sqrt{N}}\sum_{j=1}^{n}[S(V_{jN}^{**})-S(V_j)]\xrightarrow[N\to\infty]{P} 0$, from which it implies

that

$$\sqrt{N}(B_{1N}^{**}+B_{2N}^{**})-\sqrt{N}(B_{1N}+B_{2N})\xrightarrow[N\to\infty]{P} 0.$$

(2) Claim $\sqrt{N}C_{iN}^{**}\xrightarrow[N\to\infty]{P} 0$ for $i=1,2,3,4,5,6$.

The proof for the negligibility of the C^{**} - terms is briefly indicated. We first note that J(H)=H in this problem. The discussion of C_{1N}^{**} term is analogous to the argument used by Chernoff and Savage (1958) in their treatment of the term C_{1N}. To deal with C_{2N}^{**}, write

$$T_n^{**}(x) - T(x) = (T_n^{**}(x) - T^*(x)) + (T^*(x) - T(x)) \text{ and}$$

$$S_m^{**}(x) - S(x) = (S_m^{**}(x) - S^*(x)) + (S^*(x) - S(x)) \text{ where}$$

$$S^*(x) = P([(X_1 - \tfrac{t_1}{\sqrt{N}})^2 + (X_2 - \tfrac{t_2}{\sqrt{N}})^2]^{1/2} \leqslant x), \text{ and}$$

$$T^*(x) = P([(Y_1 - \tfrac{t_1}{\sqrt{N}})^2 + (Y_2 - \tfrac{t_2}{\sqrt{N}})^2]^{1/2} \leqslant x),$$

and collect the terms it can be shown as in Raghavachari (1965) that $C_{2N}^{**} = o_p(N^{-1/2})$. The terms C_{3N}^{**}, C_{4N}^{**}, C_{5N}^{**}, C_{6N}^{**} can be shown to be $o_p(N^{-1/2})$ since J(H)=H.

This completes the proof of Lemma 3.5.

Applying the technique used in Fligner (1974), we need the following lemma.

Lemma 3.6.: Choose $\delta > 0$, for any integers r and s, let

$$H_{r,s,N}(\underset{\sim}{X},\underset{\sim}{Y},\underset{\sim}{t}) = \sup_{\substack{\frac{r\delta}{\sqrt{N}} \leqslant z_1 \leqslant t_1 \\ \frac{s\delta}{\sqrt{N}} \leqslant z_2 \leqslant t_2}} |\Phi(\underset{\sim}{X} - \underset{\sim}{z}, \underset{\sim}{Y} - \underset{\sim}{z}) - \Phi(\underset{\sim}{X} - (\tfrac{r\delta}{\sqrt{N}}, \tfrac{s\delta}{\sqrt{N}}), \underset{\sim}{Y} - (\tfrac{r\delta}{\sqrt{N}}, \tfrac{s\delta}{\sqrt{N}}))|.$$

Then

(1) $E(H_{r,s,N}(\underset{\sim}{X},\underset{\sim}{Y},(\frac{(r+1)\delta}{\sqrt{N}}, \frac{(s+1)\delta}{\sqrt{N}}))) \leqslant K_1\delta/\sqrt{N}$ for some K_1, and

(2) $$E([H_{r,s,N}(\underset{\sim}{X}_i,\underset{\sim}{Y}_j,(\tfrac{(r+1)\delta}{\sqrt{N}}, \tfrac{(s+1)\delta}{\sqrt{N}})) - E(H_{r,s,N}(\underset{\sim}{X}_i, \underset{\sim}{Y}_j, (\tfrac{(r+1)\delta}{\sqrt{N}}, \tfrac{(s+1)\delta}{\sqrt{N}})))]$$
$$\cdot[H_{r,s,N}(\underset{\sim}{X}_k,\underset{\sim}{Y}_\ell,(\tfrac{(r+1)\delta}{\sqrt{N}}, \tfrac{(s+1)\delta}{\sqrt{N}})) - E(H_{r,s,N}(\underset{\sim}{X}_k,\underset{\sim}{Y}_\ell,(\tfrac{(r+1)\delta}{\sqrt{N}}, \tfrac{(s+1)\delta}{\sqrt{N}})))])$$
$$\xrightarrow[N\to\infty]{} 0 \text{ for any } i, j, k, \ell.$$

Proof: (1) Without loss of generality, we can take r = s = 0.

By definition, $$H_{o,o,N}(\underset{\sim}{X},\underset{\sim}{Y},(\tfrac{\delta}{\sqrt{N}}, \tfrac{\delta}{\sqrt{N}})) = \sup_{\substack{0 \leqslant z_1 \leqslant \delta/\sqrt{N} \\ 0 \leqslant z_2 \leqslant \delta/\sqrt{N}}} |\Phi(\underset{\sim}{X}-\underset{\sim}{z},\underset{\sim}{Y}-\underset{\sim}{z})-\Phi(\underset{\sim}{X},\underset{\sim}{Y})|.$$

We claim that if $|\,|\underset{\sim}{X}| - |\underset{\sim}{Y}|\,| > 4\delta/\sqrt{N}$ then $H_{o,o,N}(\underset{\sim}{X},\underset{\sim}{Y},(\frac{\delta}{\sqrt{N}}, \frac{\delta}{\sqrt{N}})) = 0.$

Case (a) Assume $|\underset{\sim}{X}| - |\underset{\sim}{Y}| > 4\delta/\sqrt{N}$.

$$\begin{aligned} |\underset{\sim}{X} - \underset{\sim}{z}| - |\underset{\sim}{Y} - \underset{\sim}{z}| &\geqslant (|\underset{\sim}{X}| - |\underset{\sim}{z}|) - (|\underset{\sim}{Y}| + |\underset{\sim}{z}|) \\ &= |\underset{\sim}{X}| - |\underset{\sim}{Y}| - 2|\underset{\sim}{z}| \\ &> 4\delta/\sqrt{N} - 2|\underset{\sim}{z}| \\ &> 0 \text{ if } 0 \leqslant z_1 \leqslant \delta/\sqrt{N} \text{ and } 0 \leqslant z_2 \leqslant \delta/\sqrt{N}. \end{aligned}$$

Case (b) Assume $|\underset{\sim}{Y}| - |\underset{\sim}{X}| > 4\delta/\sqrt{N}$.

Similarly, we have $|\underset{\sim}{Y} - \underset{\sim}{z}| - |\underset{\sim}{X} - \underset{\sim}{z}| > 0$ if $0 \leqslant z_1 \leqslant \delta/\sqrt{N}$ and $0 \leqslant z_2 \leqslant \delta/\sqrt{N}$.

Combining cases (a) and (b), we have shown that if

$$|\,|\underset{\sim}{X}| - |\underset{\sim}{Y}|\,| > 4\delta/\sqrt{N}, \text{ then } H_{o,o,N}(\underset{\sim}{X},\underset{\sim}{Y},(\tfrac{\delta}{\sqrt{N}},\tfrac{\delta}{\sqrt{N}})) = 0.$$

Therefore, $E(H_{o,o,N}(\underset{\sim}{X},\underset{\sim}{Y},(\frac{\delta}{\sqrt{N}},\frac{\delta}{\sqrt{N}})))$

$$\begin{aligned} &= P(H_{o,o,N}(\underset{\sim}{X},\underset{\sim}{Y},(\tfrac{\delta}{\sqrt{N}},\tfrac{\delta}{\sqrt{N}})) = 1) \\ &\leqslant P(|\,|\underset{\sim}{X}| - |\underset{\sim}{Y}|\,| \leqslant 4\delta/\sqrt{N}) \\ &\leqslant K_1\delta/\sqrt{N}, \text{ by the assumption that the p.d.f.'s of } |\underset{\sim}{X}| \text{ and } |\underset{\sim}{Y}| \text{ are bounded.} \end{aligned}$$

This completes the proof of (1)

(2) Follows easily from (1).

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: In order to show that $W^*_{m,n}$ and $W_{m,n}$ have the same limiting distribution, it is sufficient to show that

$$\sup_{|\underset{\sim}{t}| \leqslant c} |S_N(\underset{\sim}{t})| \xrightarrow[N\to\infty]{P} 0, \text{ where}$$

$$S_N(\underset{\sim}{t}) = m^{-\frac{1}{2}}n^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\{\Phi(\underset{\sim}{X}_i - \underset{\sim}{t}/\sqrt{N}, \underset{\sim}{Y}_j - \underset{\sim}{t}/\sqrt{N}) - \Phi(\underset{\sim}{X}_i, \underset{\sim}{Y}_j)\},$$

where $\Phi(\underset{\sim}{x},\underset{\sim}{y}) = 1$ if $|\underset{\sim}{x}| > |\underset{\sim}{y}|$

$= 0$ otherwise,

by the facts that

$$\frac{W^*_{m,n} - E(W_{m,n})}{\sqrt{Var(W_{m,n})}} - \frac{W_{m,n} - E(W_{m,n})}{\sqrt{Var(W_{m,n})}} = \frac{W^*_{m,n} - W_{m,n}}{\sqrt{Var(W_{m,n})}}$$

$$= \frac{m^{-\frac{1}{2}}n^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\{\Phi(\underset{\sim}{X}_i - \bar{\underset{\sim}{Z}}_N, \underset{\sim}{Y}_j - \bar{\underset{\sim}{Z}}_N) - \Phi(\underset{\sim}{X}_i, \underset{\sim}{Y}_j)\}}{m^{-\frac{1}{2}}n^{-1}\sqrt{Var(W_{m,n})}}$$

and $m^{-1/2}n^{-1}\sqrt{Var(W_{m,n})}$ is a constant for any m,n such that m/N remains constant (see Gibbons (1971), p.158–160 and p.166–167), and by Lemmas 3.2, 3.3 and 3.4

Let $\varepsilon_1, \varepsilon_2 > 0$ be given. Set $\delta = \varepsilon_1/2K_1$, where K_1 is the bound described in (1) of Lemma 3.6.

(i) Since the unit disc $\{\underset{\sim}{t}: |\underset{\sim}{t}| \leqslant c\}$ can be covered by a finite number, say M, of squares with length δ of each side, we have

$$P(\sup_{|\underset{\sim}{t}|\leqslant c} |S_N(\underset{\sim}{t})| > \varepsilon_1) \leqslant \sum_{\substack{M \text{ pairs} \\ \text{of } (r,s)}} P(\sup_{\substack{r\delta \leqslant t_1 \leqslant (r+1)\delta \\ s\delta \leqslant t_2 \leqslant (s+1)\delta}} |S_N(\underset{\sim}{t})| > \varepsilon_1).$$

(ii) For any integers r and s, we claim that there exists a positive integer $N_{r,s}$ such that

$$P(\sup_{\substack{r\delta \leqslant t_1 \leqslant (r+1)\delta \\ s\delta \leqslant t_2 \leqslant (s+1)\delta}} |S_N(\underset{\sim}{t})| > \varepsilon_1) < \varepsilon_2/M \text{ for all } N \geqslant N_{r,s}.$$

Consider $r\delta \leqslant t_1 \leqslant (r+1)\delta$ and $s\delta \leqslant t_2 \leqslant (s+1)\delta$. By definition, $S_N(\underset{\sim}{t}) = S_{r,s,N}(\underset{\sim}{t}) + S_N((r\delta, s\delta))$, where

$$S_{r,s,N}(\underset{\sim}{t}) = m^{-1/2}n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} \{ \Phi(\underset{\sim}{X}_i - \frac{\underset{\sim}{t}}{\sqrt{N}}, \underset{\sim}{Y}_j - \frac{\underset{\sim}{t}}{\sqrt{N}}) - \Phi(\underset{\sim}{X}_i - (\frac{r\delta}{\sqrt{N}}, \frac{s\delta}{\sqrt{N}}), \underset{\sim}{Y}_j - (\frac{r\delta}{\sqrt{N}}, \frac{s\delta}{\sqrt{N}})) \}.$$

Then $|S_{r,s,N}(\underset{\sim}{t})|$

$$\leqslant m^{-1/2}n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} |\Phi(\underset{\sim}{X}_i - \frac{\underset{\sim}{t}}{\sqrt{N}}, \underset{\sim}{Y}_j - \frac{\underset{\sim}{t}}{\sqrt{N}}) - \Phi(\underset{\sim}{X}_i - (\frac{r\delta}{\sqrt{N}}, \frac{s\delta}{\sqrt{N}}), \underset{\sim}{Y}_j - (\frac{r\delta}{\sqrt{N}}, \frac{s\delta}{\sqrt{N}}))|$$

$$\leqslant m^{-1/2}n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} H_{r,s,N}(\underset{\sim}{X}_i, \underset{\sim}{Y}_j, (\frac{(r+1)\delta}{\sqrt{N}}, \frac{(s+1)\delta}{\sqrt{N}}))$$

$$= m^{-1/2}n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} \{ H_{r,s,N}(\underset{\sim}{X}_i, \underset{\sim}{Y}_j, (\frac{(r+1)\delta}{\sqrt{N}}, \frac{(s+1)\delta}{\sqrt{N}}))$$

$$- E(H_{r,s,N}(\underset{\sim}{X}_i, \underset{\sim}{Y}_j, (\frac{(r+1)\delta}{\sqrt{N}}, \frac{(s+1)\delta}{\sqrt{N}}))) \}$$

$$+ m^{1/2} E(H_{r,s,N}(\underset{\sim}{X}, \underset{\sim}{Y}, (\frac{(r+1)\delta}{\sqrt{N}}, \frac{(s+1)\delta}{\sqrt{N}}))).$$

By Lemma 3.6 (1), we have

$|S_{r,s,N}(\underset{\sim}{t})| \leqslant Z_N + m^{1/2}N^{-1/2}K_1\delta \leqslant Z_N + \varepsilon_1/2$, where

$$Z_N = m^{-\frac{1}{2}}n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} \{ H_{r,s,N}(\underset{\sim}{X}_i,\underset{\sim}{Y}_j,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}}))$$

$$- E(H_{r,s,N}(\underset{\sim}{X}_i,\underset{\sim}{Y}_j,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}}))) \}.$$

Since Z_N does not depend on $r\delta \leqslant t_1 \leqslant (r+1)\delta$ and $s\delta \leqslant t_2 \leqslant (s+1)\delta$, it implies that $\sup\limits_{\substack{r\delta\leqslant t_1\leqslant(r+1)\delta \\ s\delta\leqslant t_2\leqslant(s+1)\delta}} |S_{r,s,N}(\underset{\sim}{t})| \leqslant Z_N + \varepsilon_1/2.$

Further, $E(Z_N{}^2)$

$$= m^{-1}n^{-2} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\ell=1}^{n} E([H_{r,s,N}(\underset{\sim}{X}_i,\underset{\sim}{Y}_j,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}}))$$

$$- E(H_{r,s,N}(\underset{\sim}{X}_i,\underset{\sim}{Y}_j,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}})))]$$

$$\cdot[H_{r,s,N}(\underset{\sim}{X}_k,\underset{\sim}{Y}_\ell,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}}))$$

$$- E(H_{r,s,N}(\underset{\sim}{X}_k,\underset{\sim}{Y}_\ell,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}})))])$$

and for $i \neq k$ and $j \neq \ell$,

$$E([H_{r,s,N}(\underset{\sim}{X}_i,\underset{\sim}{Y}_j,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}})) - E(H_{r,s,N}(\underset{\sim}{X}_i,\underset{\sim}{Y}_j,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}})))]$$

$$\cdot[H_{r,s,N}(\underset{\sim}{X}_k,\underset{\sim}{Y}_\ell,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}})) - E(H_{r,s,N}(\underset{\sim}{X}_k,\underset{\sim}{Y}_\ell,(\tfrac{(r+1)\delta}{\sqrt{N}},\tfrac{(s+1)\delta}{\sqrt{N}})))])$$

$= 0$.

Then $E(Z_N{}^2) \xrightarrow[N\to\infty]{} 0$ by Lemma 3.6 (2), which implies that $Z_N \xrightarrow[N\to\infty]{P} 0$ by Chebyshev's inequality.

Since $S_N(\underset{\sim}{t}) = S_{r,s,N}(\underset{\sim}{t}) + S_N((r\delta,s\delta))$, it follows that

$$\sup_{\substack{r\delta\leqslant t_1\leqslant(r+1)\delta \\ s\delta\leqslant t_2\leqslant(s+1)\delta}} |S_N(\underset{\sim}{t})| \leqslant \sup_{\substack{r\delta\leqslant t_1\leqslant(r+1)\delta \\ s\delta\leqslant t_2\leqslant(s+1)\delta}} |S_{r,s,N}(\underset{\sim}{t})| + |S_N((r\delta,s\delta))|$$

$$\leqslant Z_N + \varepsilon_1/2 + |S_N((r\delta,s\delta))|.$$

By the result that $Z_N \xrightarrow[N \to \infty]{P} 0$ and Lemma 3.5, we have

$Z_N + |S_N((r\delta, s\delta))| \xrightarrow[N \to \infty]{P} 0$, which implies that there exists a positive integer $N_{r,s}$ such that

$$P(Z_N + |S_N((r\delta,s\delta))| > \varepsilon_1/2) < \varepsilon_2/M \text{ for all } N \geq N_{r,s}.$$

Hence
$$P\Big(\sup_{\substack{r\delta \leq t_1 \leq (r+1)\delta \\ s\delta \leq t_2 \leq (s+1)\delta}} |S_N(\underset{\sim}{t})| > \varepsilon_1\Big)$$
$$\leq P(Z_N + |S_N((r\delta,s\delta))| > \varepsilon_1/2)$$
$$< \varepsilon_2/M \text{ for all } N \geq N_{r,s}.$$

(iii) Take $N_o = \max\limits_{\substack{M \text{ pairs} \\ \text{of } (r,s)}} \{ N_{r,s} \}$. From (i) and (ii), we obtain

$$P\Big(\sup_{|\underset{\sim}{t}| \leq c} |S_N(\underset{\sim}{t})| > \varepsilon_1\Big) \leq \sum_{\substack{M \text{ pairs} \\ \text{of } (r,s)}} P\Big(\sup_{\substack{r\delta \leq t_1 \leq (r+1)\delta \\ s\delta \leq t_2 \leq (s+1)\delta}} |S_N(\underset{\sim}{t})| > \varepsilon_1\Big)$$
$$< M \cdot \varepsilon_2/M = \varepsilon_2 \text{ for all } N \geq N_o.$$

This completes the proof of Theorem 3.1.

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