THE ADVANTAGE OF SECOND GUESSER IN A TWO-PERSON ZERO-SUM GAME

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摘　　要

本文研究兩人對局的遊戲。首先，雙方在不讓另一方知道的情況下，投入一任意整數值的賭金於一盒中，然後依序猜測盒內的金額總數，猜對的即可贏得賭局，取得所有賭金。本文探討後猜者的優勢，同時也使用電腦模擬驗証理論推導的結果。

Abstract

A guessing game between two persons, say, A and B, is of interest. First, without letting the other knows, A and B each put a certain amount of money into a box. The goal is to guess the total money in the box, and the one with the correct guess wins the game and takes away all the money. A guesses first, and then B. We are interested in the advantage of B over A, and simulation are performed to compare with the result derived.

1. INTRODUCTION

A guessing game, which was often played in our childhood, is of interest. The game is played between two persons, namely A and B. First, A and B must put $\$x$ and $\$y$ in a box according to their will, without letting the other person knows. The goal is to guess the sum of x and y, and the one with the correct guess takės away all the money in the box. Let A guess first, say z. After A fails, B guesses the number w, with extra information provided by the previous guess. If neither A nor B has the correct guess, the game ends. Thus, it is possible that nobody wins the game. The following is the list of notation used in this report:

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	A	B
Bet	x	y
Guess	z	w

Let us consider an example first. Suppose that $x,y \in \{1,2\}$, and the chance of betting \$1 or \$2 is equally likely. Intuitively, if A puts \$1 in the box, the only reasonable guesses are \$2 and \$3. Similarly, A would guess \$3 or \$4 if he puts \$2 in the box. Therefore, the winning probability of A is 1/2, which is independent of the money betted by A and B. And the expected gain of A is half of the expected number of dollars put by his/her opponent, i.e. \$3/4.

After A fails, B guesses with the information from A. If the wrong guessing value, z, is \$2 (or \$4), B would know that his/her opponent put \$1 (or \$2) in the box, and B shall have the right answer. Similarly, if $z = 3$ then the only possibility is $x = y$, i.e. the money betted by A and B are the same. In either case, B shall be able to guess correctly whenever A fails. Thus, the winning probability of B is equal to the failure probability of A, 1/2, And the expected gain of B is half of the expected number of money betted by A, \$3/4. The following table illustrates the above result.

Table 1. The winning chance of A, given x, y, and z

	z=2	z=3	z=3	z=4
y=1	1	0	1	0
y=2	0	1	0	1
	x=1		x=2	

Table 1 (continued) The winning chance of B, given x, y, and z

	z=2	z=3	z=3	z=4
y=0	1	0	0	1
y=2	1	0	1	0
	x=1		x=2	

Although the winning probabilities (and expected gains) of A and B are the same in the above example. The second guesser B usually have better odds than A, since B has more information than A. Instead of assuming $x,y \in \{1,2\}$, let $x,y \in \{1,2,3\}$. Then the winning probability of A is 1/3. The winning probability of B can be computed similar to the above example, and is shown in Table 2.

Table 2. The winning chance of B, given x,y, and z

	z=2	Z=3	Z=4	Z=3	z=4	z=5	z=4	z=5	z=6
y=1	0	1	1/2	0	1/2	1/2	0	1/2	1
y=2	1	0	1/2	1	0	1	1/2	1	0
y=3	1	1/2	0	1/2	1/2	0	1/2	1	0
		x=1			x=2			x=3	

Therefore the winning probability of B is $(4.5+4+4.5)/27=13/27$, which is larger than 1/3 of A. Similar computation gives the expected gains of A and B, as \$2/3 and \$26/27, respectively.

The computation is the same for $x,y \in \{1,2,3,4,5,6\}$, which gives the winning probabilities of A and B, as 1/6 and 8/27, and the advantage of B over A is even larger. This matches to our intuition, and it seems that the advantage of B over A is getting larger as the range of possible bets getting larger. The general form of winning probabilities and expected gains of A and B will be discussed in the following section.

2. MAIN RESULTS

Some special cases, when the range of bets is small, is described in the preceding section. In this section, the general case will be discussed, and the exact forms of winning probabilities as well as expected gains for each guesser will be derived. Let X and Y be the bets of A and B, respectively, and Z and W be the guesses of A and B. Also, define (A wins) and (B wins) as the events that A and B win the game, and define (A gains) and (B gains) as the money that A and B gain from the game. Further, we assume that it is equally likely for A and B to bet any amount of money, i.e. X and Y take value from $1,2,\cdots,k$ each with probability $1/k$. Based on this assumption, it is natural for A to guess every possible combination of $X+Y$ with equal probability, or

$$P(Z=z \mid X=x) \frac{1}{k}, \qquad z=x+1,\ x+2,\ \ldots,\ x+k.$$

This distribution of W is more complicated and will be discussed later. Note that in our study, we also assume that the bets and guesses of A and B are "rational", and neither A or B has number preference. In other words, no prior experience of betting or guessing can be carried over to the current game.

First, we compute the winning probability of A. Clearly, no information is available for A, and intuitively, since the only information is that Y takes value from $1,2,\cdots,k$, A would guess from $x+1,x+2,\cdots x+k$. Therefore, the computation of the winning probability of A is straightforwards:

$$P(\text{A wins}) = \sum_{y=1}^{k} P\ (Y=y)P(A \text{ wins} \mid Y=y)=\frac{1}{k}. \tag{1}$$

Similarly, the expected gain of A is

$$E(A \text{ gains}) = \sum_{y=1}^{k} y \times P(Y=y)P(A \text{ wins} \mid Y=y) = \frac{k+1}{2k}. \tag{2}$$

It should be noted that the expected gain of A is just the winning probability of A times the expected bet of B in this case.

The calculation for B is more complicated. Suppose that A fails (that is, $z \neq x+y$). Then B has better odds since the possible bets of A has been reduced to the interval: $max\ \{1,z-k\} \leq x \leq min\ \{k,z-1\}$. Especially, we can divide this interval into two cases: $2 \leq z \leq k$ and $k+1 \leq z \leq 2k$, each case leads to $1 \leq x \leq z-1$ and $z-k \leq x \leq k$, or equivalently, $y+1 \leq w \leq y+z-1$ and $y+z-k \leq w \leq y+k$. We first consider the special case when $z=2$ and $z=2k$, since the computation is different. Because $z=2$ would lead to the result that A bet \$1, B could guess $x+y$ correctly if A fails. Similarly, $z=2k$ leads to $x=k$.

For the general case, we need to consider if z is a valid guess, that is, z satisfies the condition: $y+max\{1,z-k\} \leq z \leq y+min\{k,z-1\}$. For example, suppose that $3\ \leq z \leq k$ and z is a valid guess, then the possible bets of A would lie in $1 \leq x \leq z-1$ excluding the case that $x=z-y$, and with these conditions, the winning chance of B should be $1/(z-2)$. On the contrary, if z is not a valid guess, the possible bets of A would still lie in $1 \leq x \leq z-1$, and the winning chance of B should be $1/(z-1)$. Following this rule, the conditional probability of W, given $Y=y$ and $Z=z$, is

$$P(W=w \mid Y=y,\ Z=z)$$

$$= \begin{cases} 1, & \text{if } z=2 \quad \text{and } 2 \leq y \leq k \\ \frac{1}{z-2}, & \textit{if } 3 \leq z \leq k \quad \text{and } 1 \leq y \leq z-1 \\ \frac{1}{z-1}, & \text{if } 3 \leq z \leq k \quad \text{and } z \leq y \leq k \\ \frac{1}{2k-z+1}, & \text{if } k+1 \leq z \leq 2k-1 \quad \text{and } 1 \leq y \leq z-k-1 \\ \frac{1}{2k-z}, & \text{if } k+1 \leq z \leq 2k-1 \quad \text{and } z-k \leq y \leq k \\ 1, & \text{if } z=2k \quad \text{and } 1 \leq y \leq k-1. \end{cases}$$

And the winning probability of B can thus be formularized as:

$$P(B \text{ wins})$$

$$= \sum_{x=1}^{k} P(X=x) \sum_{y=1}^{k} P(Y=y) \sum_{z=z+1}^{x+k} P(Z=z \mid X=x)$$

$$\times P(B \text{ wins} \mid X=x,\ Y=y,\ Z=z) I[z \neq x+y]$$

$$= \sum_{x=1}^{k} P(X=x) \sum_{y=1}^{k} P(Y=y) \sum_{z=x+1}^{x+k} P(Z=z \mid X=x)$$

$$\times \{ \sum_{\forall w} P(W=w \mid Y=y,\ Z=z) I[w=x+y] \} I[z \neq x+y]$$

$$= \sum_{z=2}^{k} \sum_{x=1}^{z-1} \{ \sum_{y=1}^{z-1} P(X=x, Z=z) P(Z=z) P(Y=y) \frac{1}{z-2} I[z \neq x+y]$$

$$+ \sum_{y=z}^{k} P(X=x \mid Z=z) P(Z=z) P(Y=y) \frac{1}{z-1} \}$$

$$+ \sum_{z=k+1}^{2k} \sum_{x=z-k}^{k} \{ \sum_{y=1}^{z-k-1} P(X=x \mid Z=z)P(Z=z)P(Y=y) \frac{1}{2k-z+1}$$

$$+ \sum_{y=z-k}^{k} P(X=x \mid Z=z)P(Z=z)P(Y=y) \frac{1}{2k-z} I[z \neq x+y] \} ,$$

where $I[\bullet]$ is the indicator function, and the third equation above follows from interchange and partition of the summations. Note that the above equation can be applied to different assumptions of $X,Y,Z,$ and $W,$ although they are assumed to be uniformly distributed.

To compute $P(B$ wins$)$, we also start with the cases $z = 2$ and $z = 2k$. Since $z = 2$ and $z = 2k$ would lead to the perfect information for B, when A fails, we have

$$P((B \text{ wins}) \cap (Z=2)) = P(Z=2,\ X=1) \sum_{y=2}^{k} P(Y=y) = \frac{k-1}{k^3}$$

$$P((B \text{ wins}) \cap (Z=2k)) = P(Z=2k,\ X=k) \sum_{y=1}^{k-1} P(Y=y) = \frac{k-1}{k^3}$$

When $3 \leq z \leq k$,

$$P((B \text{ wins}) \cap (Z=z))$$

$$= \sum_{y=1}^{z-1} P(Y=y) \frac{1}{z-2} \sum_{x=1}^{z-1} P(X=x \mid Z=z)P(Z=z)I[x \neq z-y]$$

$$+ \sum_{y=z}^{k} P(Y=y) \frac{1}{z-1} \sum_{x=1}^{z-1} P(X=x \mid Z=z)P(Z=z)$$

$$= \sum_{y=1}^{z-1} \frac{1}{k} \times \frac{1}{z-2} \times \frac{z-2}{k^2} + \sum_{y=z}^{k} \frac{1}{k} \times \frac{1}{z-1} \times \frac{z-1}{k^2}$$

$$= \frac{1}{k^2} .$$

When $k+1 \leq z \leq 2k-1$,
$P((B \text{ wins}) \cap (Z=z))$

$$= \sum_{y=1}^{z-k-1} P(Y=y) \frac{1}{2k-z+1} \sum_{x=z-k}^{k} P(X=x \mid Z=z)P(Z=z)$$

$$+ \sum_{y=z-k}^{k} P(Y=y) \frac{1}{2k-z} \sum_{x=z-k}^{k} P(X=x \mid Z=z)P(Z=z)I[x \neq z-y]$$

$$= \sum_{y=1}^{z-k-1} \frac{1}{k} \times \frac{1}{2k-z+1} \times \frac{2k-z+1}{k^2} + \sum_{y=z}^{k} \frac{1}{k} \times \frac{1}{2k-z} \times \frac{2k-z}{k^2}$$

$$= \frac{1}{k^2} .$$

So, the winning probability of B is

$$\begin{aligned} P(B \text{ wins}) &= \sum_{z=2}^{2k} P((B \text{ wins}) \cap (Z=z)) \\ &= 2 \times \frac{k-1}{k^3} + \frac{1}{k^2} \times (2k-3) \\ &= \frac{2k^2-k-2}{k^3} . \end{aligned} \tag{3}$$

The calculation of the expected gain of B is similar to that of the winning probability of B. And we still separate the expected gain by z's. First, for $z=2$ and $z=2k$.

$$E((B \text{ gains}) \cap (Z=2)) = P(Z=2,\ X=1) \sum_{y=2}^{k} P(Y=y) = \frac{k-1}{k^3}$$

$$E((B \text{ gains}) \cap (Z=2k)) = k \times P(Z=2k,\ X=k) \sum_{y=1}^{k-1} P(Y=y) = \frac{k-1}{k^3}$$

When $3 \leq z \leq k$,

$E((B \text{ gains}) \cap (Z=z))$

$$= \sum_{y=1}^{z-1} P(Y=y) \frac{1}{z-2} \sum_{x=1}^{z-1} x \times P(X=x \mid Z=z)P(Z=z)I[x \neq z-y]$$

$$+ \sum_{y=z}^{k} P(Y=y) \frac{1}{z-1} \sum_{x=1}^{z-1} x \times P(X=x \mid Z=z)P(Z=z)$$

$$= \sum_{y=1}^{z-1} \frac{1}{k} \times \frac{1}{z-2} \times \frac{1}{k^2} \left\{ \frac{z(z-3)}{2} + y \right\}$$

$$+ \sum_{y=z}^{k} \frac{1}{k} \times \frac{1}{z-2} \times \frac{1}{k^2} \left\{ \frac{z(z-1)}{2} + y \right\}$$

$$= \frac{z}{2k^2}.$$

Finally, when $k+1 \leq z \leq 2k-1$,

$E((B \text{ gains}) \cap (Z=z))$

$$= \sum_{y=1}^{z-k-1} P(Y=y) \frac{1}{2k-z+1} \sum_{x=z-k}^{k} x \times P(X=x \mid Z=z)P(Z=z)$$

$$+ \sum_{y=z-k}^{k} P(Y=y) \frac{1}{2k-z} \sum_{x=z-k}^{k} x \times P(X=x \mid Z=z)P(Z=z)I[x \neq z-y]$$

$$= \sum_{y=1}^{z-k-1} \frac{1}{k} \times \frac{1}{2k-z+1} \times \frac{1}{k^2} \left\{ \frac{z(2k-z+1)}{2} \right\}$$

$$+ \sum_{y=z-k}^{k} \frac{1}{k} \times \frac{1}{2k-z} \times \frac{1}{k^2} \left\{ \frac{z(2k-z-1)}{2} + y \right\}$$

$$= \frac{z}{2k^2}.$$

So, the expected gain of B is

$$E(B \text{ gains}) = \frac{k-1}{k^2} + \sum_{z=3}^{2k-1} \frac{z}{2k^2} + \frac{k-1}{k^2}$$

$$= \frac{2k^3+k^2-3k-2}{2k^3} \text{ or } \frac{(k+1)(2k^2-k-2)}{2k^3}. \tag{4}$$

As a check, we plug into $k=2$, $P(B \text{ wins}) = 1/2$ and $E(B \text{ gains}) = 3/4$, which matches to our previous result.

3. DISCUSSION AND SIMULATION

It should be noted that $k \times P(A \text{ wins}) \to 1$ and $k \times P(B \text{ wins}) \to 2$ as $k \to \infty$, and also, $E(A \text{ gains}) \to 1/2$ and $E(B \text{ gains}) \to 1$ as $k \to \infty$. This suggests that when k is large, the winning probabilities of A and B are of the order of $1/k$, and the expected gains of A and B will converge to 1/2 and 1, respectively. The advantage of second guesser, B, is about twice as much as the first guesser, A, in both the winning probability and the expected gain, as k gets bigger.

To verify the above results, we used simulation as a check. Table 3 lists the simulations of $k = 2,3,6,10,100$ and each with 10,000 of replications. The top numbers in each cell are the simulated values and the numbers inside the parenthesis are the theoretical values. The winning probabilities and expected gains of A and B are all closed to the theoretical values.

Table 3. Simulations for various k

	k=2	k=3	k=6	k=10	k=100
P(A wins)	0.4974 (0.5)	0.3361 (0.333)	0.1710 (0.1667)	0.1014 (0.1)	0.0096 (0.01)
P(B wins)	0.5026 (0.5)	0.4815 (0.4815)	0.2958 (0.2963)	0.1867 (0.188)	0.0163 (0.0199)
E(A gains)	0.7411 (0.75)	0.6712 (0.6667)	0.6058 (0.5833)	0.5454 (0.55)	0.5529 (0.505)
E(B gains)	0.7559 (0.75)	0.9423 (0.963)	1.0346 (1.037)	0.9957 (1.034)	0.9208 (1.0048)

Suppose, instead of the uniform guess, we are interested in the guess of A which would minimize the winning probability of B. Since for every z the chance that B wins is

$$P(B \text{ wins} \mid Z=z) = \begin{cases} \frac{k-1}{k}, & \text{if } z = 2 \text{ or } 2k \\ \frac{1}{z-1}, & \text{if } 3 \le z \le k \\ \frac{1}{2k-z+1}, & \text{if } k+1 \le z \le 2k-1 \end{cases}$$

A should guess $z=k+1$ to minimize the winning probability of B, no matter what A bets. And since this is true for $x = 1,2,\ldots,k$, A would let x as small as possible to reduce the payoff of B. Therefore we would suggest that the best strategy of A, under the assumption that Y and W are uniformly distributed, is $X=1$ and $Z=k+1$ with probability 1. And this strategy of A will result in $P(A \text{ wins}) = P(B \text{ wins}) = 1/k$ and

$$E(A \text{ gains}) = \frac{1}{k} \times k = 1,$$

$$E(B \text{ gains}) = \frac{1}{k} \times 1 = \frac{1}{k}.$$

Of course, if B knows that A does not guess uniformly, the advantage of A in the expected gain would disappear.

For the other extreme, suppose that X and Y are still uniformly distributed and $Z = k+1$ with probability 1. Let

$$W = \begin{cases} k, & \text{if } y=1 \\ k+y, & \text{if } 2 \leq y \leq k. \end{cases}$$

Then $P(A \text{ wins})$ and $P(B \text{ wins})$ are still $1/k$. But

$$E(A \text{ gains}) = \frac{1}{k} \times \frac{k+1}{2} = \frac{k+1}{2k},$$

and

$$E(B \text{ gains}) = \frac{1}{k^2} \times (k-1) + \sum_{y=2}^{k} \frac{1}{k^2} \times k = \frac{k^2-1}{k^2}. \qquad (5)$$

The difference of the expected gains of B between (4) and (5) is converging to 1/2 as $k \rightarrow \infty$. Hence, even though $P(A \text{ wins}) = P(B \text{ wins}) = 1/k$, there is no significant advantage of A guessing $z = k+1$ always over guessing uniformly, if B knows what the guessing strategy that A uses.

For the general case, if X and Y are uniformly distributed with $X+1 \leq Z \leq X+k$, $Y+max\ \{1,Z-k\} \leq W \leq Y+min\ \{k,Z-1\}$, and $W \neq Z$, it can be shown that $P(A \text{ wins}) = 1/k$ and $P(B \text{ wins}) = 1/k$. However, the expected gains of A and B would be much more different, and they shall be dictated by the guessing strategies of A and B if neither A nor B knows what strategy used by each other.

4. COMMENTS

From (1) through (4), the winning probability of B is about twice as much as that of A. The comparison of expected gains between A and B is similar. It would be fair to double the expected gain of A to reach the balance. On the other hand, another seemingly fair alternative is to let A guess twice and B just once. Of course, under this setting, the winning probability and the expected gain of A will be doubled, but B will have more information, too. From our simulation study, we found that B still has better odds. And even we increase the number of guesses of A, B still has larger winning probability, provided that k is large enough. We will explore this more in the future.

Because this study is under the assumptions that both the bets are uniformly distributed and the guesses are not only rational but also uniformly distributed, interested readers could discuss the possible outcomes for changing any of the assumptions. Furthermore, this study can also be extended to more than two players, which we will also consider in the future.

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